MAT 319 - Foundations of Analysis

Midterm II Practice Problems

Stony Brook University
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Sequences and Series

(i) State the Bolzano–Weierstrass Theorem.

(ii) Does there exist a sequence that is unbounded, but has a convergent subsequence? If not, then provide a proof. If yes, give an example and prove your claim.

(iii) Give the definition of a Cauchy sequence.

(iv) Prove the following statement or provide a counterexample. Suppose that (x_n) is a sequence with the property that $\lim(x_{n+1} - x_n) = 0$. Then (x_n) is a Cauchy sequence.

(v) Show directly from the definition that the following sequence is not a Cauchy sequence.

$$x_n = \begin{cases} 2, & n \text{ even} \\ 5, & n \text{ odd} \end{cases}$$

(vi) If (x_n) is an unbounded sequence with $x_n > 0$ for each $n \in \mathbb{N}$, show that it has a subsequence (x_{n_k}) with $\lim x_{n_k} = \infty$.

(vii) Show that if $x_n > 0$ for each $n \in \mathbb{N}$, then $\lim x_n = 0$ if and only if $\lim(1/x_n) = \infty$.

(viii) Let (x_n) be a sequence. Give the definition of convergence of the series $\sum_{n=1}^{\infty} x_n$.

(ix) Suppose that $\sum_{n=1}^{\infty} x_n$ and $\sum_{n=1}^{\infty} y_n$ converge. Show that $\sum_{n=1}^{\infty}(x_n + y_n)$ converges.

(x) Determine whether the following series is convergent or divergent. Justify carefully your claims using some convergence or divergence criteria.

$$\sum_{n=1}^{\infty} \frac{5\sin^2 n}{\sqrt{4^n + 2n}}$$

(xi) Determine whether the following series is convergent or divergent. Justify carefully your claims using some convergence or divergence criteria.

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{2n-1}}$$

(xii) Is it true in general that if the series $\sum_{n=1}^{\infty} x_n^3$ converges and $x_n \geq 0$, then the series $\sum_{n=1}^{\infty} \sqrt{x_n}$ also converges? If yes, then provide a proof. Otherwise, give a counterexample.

(xiii) Suppose that (x_n) is a sequence of positive numbers such that $\sum_{n=1}^{\infty} \sqrt{x_n}$ is convergent. Then prove that the series $\sum_{n=1}^{\infty} x_n^3$ is convergent. You can use a theorem that we proved in class without proof.

Functions

(i) Let $f\colon \mathbb{R} \to \mathbb{R}$ be a function and $c \in \mathbb{R}$. When do we say that $\lim_{x\to c} f(x) = L$?

(ii) Using the definition of the limit **only**, show that

$$\lim_{x\to -1} \frac{x+5}{2x+3} = 4.$$

(iii) Using the definition of the limit **only**, show that

$$\lim_{x\to 1} \sqrt{x} = 1.$$

(iv) If $f(x) = \sin(1/x)$, $x \neq 0$, then show that $\lim_{x\to 0} f(x)$ does not exist. If $g(x) = x\sin(1/x)$, $x \neq 0$, then show that $\lim_{x\to 0} g(x)$ exists.

(v) Let $f\colon \mathbb{R} \to \mathbb{R}$ be a function and $c \in \mathbb{R}$. When do we say that $\lim_{x\to c} f(x) = \infty$?

(vi) Let $g\colon (c,\infty) \to \mathbb{R}$ be a function with $g(x) \neq 0$ for all $x \in (c,\infty)$. Using the definition of the limit, show that

$$\text{if} \quad \lim_{x\to c} g(x) = \infty \quad \text{then} \quad \lim_{x\to c} \frac{1}{g(x)} = 0.$$

(vii) Consider the piecewise defined function

$$h(x) = \begin{cases} 2x^2+1, & x \text{ rational} \\ x+1, & x \text{ irrational} \end{cases}.$$

Prove that the function h is continuous at $c = 0$ and $c = 1/2$, and discontinuous at $c = 1$. You may use the definition of continuity or some (dis)continuity criteria. Then show that the function h is not continuous at any $c \notin \{0, 1/2\}$.

(viii) Show that the function $f(x) = |x|$ is continuous at every $c \in \mathbb{R}$.

(ix) Show that the function $\sqrt{x}$ is continuous at every $c \geq 0$. (Hint: treat separately the cases $c = 0$ and $c > 0$.)

(x) Let $f\colon \mathbb{R} \to \mathbb{R}$ be a function such that $|f(x) - f(y)| \leq 3|x-y|$ for each $x, y \in \mathbb{R}$. Show that f is continuous at every $c \in \mathbb{R}$.

(xi) If $f\colon [0,2] \to \mathbb{R}$ is continuous and $f(1) > 0$, show that there exists $\delta > 0$ such that $f(x) > 0$ for all $x \in (1-\delta, 1+\delta)$.